Lecture 23: Cup and Cap product

One of the key structure that distinguishes cohomology with homology is that cohomology carries an algebraic structure so $\mathrm{H}^\bullet(X)$ becomes a ring. This algebraic structure is called cup product. Moreover, $\mathrm{H}_\bullet(X)$ will be a module of $\mathrm{H}^\bullet(X)$, and this module structure is called cap product.

Let R be a commutative ring with unit. We have cochain maps

$$S^\bullet(X;R)\otimes_R S^\bullet(Y;R)\to \mathrm{Hom}(S_\bullet(X)\otimes S_\bullet(Y),R)\to S^\bullet(X\times Y;R)$$

- the first map sends $\varphi_p\in S^p(X;R),\eta_q\in S^q(X;R)$ to

  $$\varphi_p\otimes\eta_q:\sigma_p\otimes\sigma_q\to\varphi_p(\sigma_p)\cdot\eta_q(\sigma_q),\quad \sigma_p\in S_p(X),\quad \sigma_q\in S_q(X).$$

- the second map is dual (applying $\mathrm{Hom}(-,R)$) to the Alexander-Whitney map

  $$AW: S_\bullet(X\times Y)\to S_\bullet(X)\otimes S_\bullet(Y).$$

This leads to a cochain map

$$S^\bullet(X; R) \otimes_R S^\bullet(Y; R) \to S^\bullet(X \times Y; R)$$

which further induces

$$\mathrm{H}^\bullet(X; R) \otimes_R \mathrm{H}^\bullet(Y; R) \to \mathrm{H}^\bullet(X \times Y; R).$$

Cup product

Definition

Let R be a commutative ring with unit. We define the cup product on cohomology groups

$$\cup : \mathrm{H}^p(X;R)\otimes_R \mathrm{H}^q(X;R)\to \mathrm{H}^{p+q}(X;R)$$

by the composition

$$\begin{tikzcd}
\mathrm{H}^\bullet(X;R)\otimes_R \mathrm{H}^\bullet(X;R) \arrow[r] \arrow[dr, "\cup"'] & \mathrm{H}^\bullet(X\times X;R) \arrow[d, "\Delta^*"] \\
 & \mathrm{H}^\bullet(X;R)
\end{tikzcd}$$

Here $\Delta : X\to X\times X$ is the diagonal map.

Alexander-Whitney map gives an explicit product formula

$$(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma) \cdot \beta(\sigma_q)$$

for

$$\alpha \in S^p(X; R), \beta \in S^q(X; R), \sigma : \Delta^{p+q} \to X.$$

Theorem

$\mathrm{H}^\bullet(X; R)$ is a graded commutative ring with uint:

1. Unit: let $1 \in \mathrm{H}^0(X; R)$ be represented by the cocyle which takes every singular 0-simplex to $1 \in R$. Then

$$1 \cup \alpha = \alpha \cup 1 = \alpha, \quad \forall \alpha \in \mathrm{H}^\bullet(X; R).$$

2. Associativity:

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma).$$

3. Graded commutativity:

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha, \quad \forall \alpha \in \mathrm{H}^p(X; R), \beta \in \mathrm{H}^q(X; R).$$

Proof

One approach is to check explicitly using Alexander-Whitney map. We give a formal proof using Eilenberg-Zilber Theorem.

First we observe that the following two compositions of Eilenberg-Zilber maps are chain homotopic

$$S_\bullet(X\times Y\times Z)\to S_\bullet(X\times Y)\otimes S_\bullet(Z)\to S_\bullet(X)\otimes S_\bullet(Y)\otimes S_\bullet(Z)$$
$$S_\bullet(X\times Y\times Z)\to S_\bullet(X)\otimes S_\bullet(Y\times Z)\to S_\bullet(X)\otimes S_\bullet(Y)\otimes S_\bullet(Z).$$

The proof is similar to Eilenberg-Zilber Theorem.

Associativity follows from the commutative diagram

$$
\begin{tikzcd}
\mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(X\times X)\otimes\mathrm{H}^\bullet(X) \arrow[r, "(\Delta\times 1)^*"] \arrow[d] & \mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X) \arrow[d] \\
\mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X\times X) \arrow[r] \arrow[d, "(1\times\Delta)^*"] & \mathrm{H}^\bullet(X\times X\times X) \arrow[r, "(\Delta\times 1)^*"] \arrow[d, "(1\times\Delta)^*"'] & \mathrm{H}^\bullet(X\times X) \arrow[d, "\Delta^*"] \\
\mathrm{H}^\bullet(X)\otimes\mathrm{H}^\bullet(X) \arrow[r] & \mathrm{H}^\bullet(X\times X) \arrow[r, "\Delta^*"] & \mathrm{H}^\bullet(X)
\end{tikzcd}
$$

Graded commutativity follows from the fact that the interchange map of tensor product of chain complexes

$$\begin{aligned} T: C_\bullet \otimes D_\bullet &\to D_\bullet \otimes C_\bullet \\ c_p \otimes d_q &\to (-1)^{pq} d_q \otimes c_p \end{aligned}$$

is a chain isomorphism. Therefore the two chain maps

$$S_\bullet(X \times Y) \to S_\bullet(Y \times X) \to S_\bullet(Y) \otimes S_\bullet(X)$$

$$S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y) \xrightarrow{T} S_\bullet(Y) \otimes S_\bullet(X)$$

are chain homotopic, again by the uniqueness in Eilenberg-Zilber Theorem.

Set $Y = X$ we find the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] \arrow[d, "T"] & \mathrm{H}^\bullet(X \times X) \arrow[d, "="] \\
\mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(X) \arrow[r] & \mathrm{H}^\bullet(X \times X).
\end{tikzcd}$$

which gives graded commutativity. $\square$

Theorem

Let $f: X \to Y$ be a continuous map. Then

$$f^* : \mathrm{H}^\bullet(Y; R) \to \mathrm{H}^\bullet(X; R)$$

is a morphism of graded commutative rings, i.e.

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

In other words, $\mathrm{H}^\bullet(-)$ defines a functor from the category of topological spaces to the category of graded commutative rings.

Proof.

The theorem follows from the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y. \end{tikzcd}$$

$\square$

Theorem (Künneth formula)

Assumem R is a PID, and $\mathrm{H}_i(X; R)$ are finitely generated R-module, then there exists a split exact sequence of R-modules

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(X; R) \otimes \mathrm{H}^q(Y; R) \to \mathrm{H}^n(X \times Y; R) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(\mathrm{H}^p(X; R), \mathrm{H}^q(Y; R)) \to 0.$$

In particular, if $\mathrm{H}^\bullet(X; R)$ or $\mathrm{H}^\bullet(Y; R)$ are free R-modules, we have an isomorphism of graded commutative rings

$$\mathrm{H}^\bullet(X \times Y; R) \simeq \mathrm{H}^\bullet(X; R) \otimes_R \mathrm{H}^\bullet(Y; R).$$

Example

$\mathrm{H}^{\bullet}(S^n) = \mathbb{Z}[\eta]/\eta^2$ where $\eta \in \mathrm{H}^n(S^n)$ is a generator.

Example

Let $T^n = S^1 \times \cdots \times S^1$ be the n-torus. Then

$$\mathrm{H}^\bullet(T^n) \simeq \mathbb{Z}[\eta_1, \cdots, \eta_n], \quad \eta_i\eta_j = -\eta_j\eta_i$$

is the exterior algebra with n generators. Each η_i corresponds a generator of $\mathrm{H}^1(S^1)$.

Proposition

$\mathrm{H}^\bullet(\mathbb{CP}^n) = \mathbb{Z}[x]/x^{n+1}$, where $x \in \mathrm{H}^2(\mathbb{CP}^n)$ is a generator.

Proof: We prove by induction n. We know that

$$\mathrm{H}^k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 2m \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Let x be a generator of $\mathrm{H}^2(\mathbb{CP}^n)$. We only need to show that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{CP}^n)$ for each $k \leq n$.

Using cellular chain complex, we know that for $k < n$

$$\mathrm{H}^{2k}(\mathbb{CP}^n) \to \mathrm{H}^{2k}(\mathbb{CP}^k)$$

is an isomorphism. By induction, this implies that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{CP}^n)$ for $k < n$. Poincare duality theorem (which will be proved in the next section) implies that

$$\mathrm{H}^2(\mathbb{CP}^n) \otimes \mathrm{H}^{2n-2}(\mathbb{CP}^n) \xrightarrow{\cup} \mathrm{H}^{2n}(\mathbb{CP}^n)$$

is an isomorphism. This says that x^n is a generator of $\mathrm{H}^{2n}(\mathbb{CP}^n)$. This proves the proposition. $\square$

Cap product

Definition

We define the evaluation map

$$\langle -, - \rangle : S^\bullet(X; R) \times_R S_\bullet(X; R) \to R$$

as follows: for $\alpha \in S^p(X; R), \sigma \in S_p(X), r \in R$,

$$\langle \alpha, \sigma \otimes r \rangle := \alpha(\sigma) \cdot r.$$

The evaluation map is compatible with boundary map and induces an evaluation map

$$\langle -, - \rangle : \mathrm{H}^p(X; R) \otimes_R \mathrm{H}_p(X; R) \to R.$$

This generalizes to

$$S^\bullet(X;R)\otimes_R S_\bullet(X\times Y;R) \to S^\bullet(X;R)\otimes_R S_\bullet(X;R)\otimes_R S_\bullet(Y;R) \overset{\langle -,-\rangle\otimes 1}{\to} S_\bullet(Y;R)$$

which induces

$$\mathrm{H}^p(X;R)\otimes_R \mathrm{H}_{p+q}(X\times Y;R) \to \mathrm{H}_q(Y;R).$$

Definition

We define the cap product

$$\cap : \mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X;R) \to \mathrm{H}_q(X;R)$$

by the composition

$$\begin{tikzcd}
\mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X;R) \arrow[r, "1\otimes\Delta"] \arrow[dr, "\cap"'] & \mathrm{H}^p(X;R) \otimes \mathrm{H}_{p+q}(X\times X;R) \arrow[d] \\
 & \mathrm{H}_q(X;R)
\end{tikzcd}$$

Theorem

The cap product gives $\mathrm{H}_\bullet(X; R)$ a structure of $\mathrm{H}^\bullet(X; R)$-module.

Theorem

The cap product extends naturally to the relative case: for any pair $A \subset X$

$$\cap : \mathrm{H}^p(X, A) \otimes \mathrm{H}_{p+q}(X, A) \to \mathrm{H}_q(X)$$
$$\cap : \mathrm{H}^p(X) \otimes \mathrm{H}_{p+q}(X, A) \to \mathrm{H}_q(X, A)$$

Proof

Since $S^\bullet(X, A) \subset S^\bullet(X)$, we have

$$\cap : S^\bullet(X, A) \times S_\bullet(X) \to S_\bullet(X).$$

We model the cap product via the Alexander-Whitney map. Then

$$\cap : S^\bullet(X, A) \times S_\bullet(A) \to 0.$$

Therefore $\cap$ factors through

$$\cap : S^\bullet(X, A) \times \frac{S_\bullet(X)}{S_\bullet(A)} \to S_\bullet(X).$$

Passing to homology (cohomology) we find the first cap product

. The second one is proved similarly using

$$\cap : S^{\bullet}(X) \times \frac{S_{\bullet}(X)}{S_{\bullet}(A)} \to \frac{S_{\bullet}(X)}{S_{\bullet}(A)}.$$

$\square$